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Vector orthogonal relations. Vector QD-algorithm

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Abstract: Vector-Padé approximants to a function $\mathbb{F}=(f_1,\ldots,f_d)$ from $\mathbb{C}$ to $\mathbb{C}^d$ have been defined, uniquely, without any auxiliary choice than the degrees of the numerator and the denominator (the same for all the components f_i), as in the scalar case [1,5]. The denominators are associated to polynomials P_r^s, which are given by vector orthogonal properties (R) and which satisfy for each s, recurrence relations of order $d+1$ (i.e. with $d+2$ terms), called relations (D).

We study here consequences of (R) and (D): first we prove an algorithm similar to the generalized MNA-algorithm; then we define a vector QD-algorithm which links two diagonals $(P_r^s)_r$ and $(P_r^{s+1})_r$.

Conversely if a family $(P_r)_{r\geqslant 0}$ verifying (D) is given, it is possible to build $(P_r^s)_{r\geqslant 0, s\geqslant 0}$, and d linear functionals C^α, $\alpha=1,\ldots,d$, such that $P_r^0=P_r$ and (P_r^s) verify the orthogonal relations (R), with respect to the C^α.

Keywords: Padé approximants, orthogonal polynomials, QD-algorithm.

Introduction

The monic polynomials P_r^s, we consider in the sequel, have been previously defined [5] as associated to the denominators of vector Padé approximants $\tilde{P}_r^s$ ($d^0 P_r^s = r$, $\tilde{P}_r^s(x)=x^r P_r^s(x^{-1})$) of a function

$$\mathbb{F}=(f_1,\ldots,f_d)^{\mathrm{T}}, \quad f_i\colon \mathbb{C}\to\mathbb{C}.$$

$\mathbb{F}$ is supposed to be analytic in a neighbourhood of zero:

$$\mathbb{F}(t)=\sum_{i\geqslant 0}\Gamma_i t^i, \qquad \Gamma_i\in\mathbb{C}^d, \quad \Gamma_i=(C_i^\alpha)_{\alpha=1,\ldots,d}.$$

Let Γ be the linear functional

$$\Gamma(x^i)=\Gamma_i. \tag{1}$$

Γ has components C^α,

$$C^\alpha(x^i)=C_i^\alpha.$$

With these notations the P_r^s are defined by the following equations:

$$(\mathrm{R}) \quad \begin{array}{ll} (\mathrm{R}.1) \\ (\mathrm{R}.2) \\ (\mathrm{R}.3) \end{array} \begin{cases} r=nd+k, \quad 0\leqslant k<d; \\ \Gamma(P_r^s\cdot x^i)=0, \quad i=s,\ldots,s+n-1; \\ c^\alpha(P_r^s x^{n+s})=0, \quad \alpha=1,\ldots,k. \end{cases}$$

They exist and are unique if the determinant of the system {R.2, R.3} is nonzero.

$$H_r^s = \begin{vmatrix} \Gamma_s & \cdots & \Gamma_{s+r-1} \\ \vdots & & \vdots \\ \Gamma_{s+n}^{(k)} & \cdots & \Gamma_{s+r+n-1}^{(k)} \end{vmatrix} \neq 0. \tag{2}$$

In this determinant, each row $(\Gamma_s, \ldots, \Gamma_{s+r-1})$ represents d scalar rows, except the last one, which represents the first k components of $(\Gamma_{s+n}, \ldots, \Gamma_{s+r+n-1})$. An immediate consequence of (R) and (2) is an expression of P_r^s as a ratio of two determinants

$$P_r^s(x) = H_r^s(x)/H_r^s$$

$$= \begin{vmatrix} \Gamma_s & \cdots & \Gamma_{s+r} \\ \vdots & & \vdots \\ \Gamma_{s+n-1} & \cdots & \Gamma_{s+r+n-1} \\ \Gamma_{s+n}^{(k)} & \cdots & \Gamma_{s+r+n}^{(k)} \\ 1 & \cdots & x^r \end{vmatrix} \times \begin{vmatrix} \Gamma_s & \cdots & \Gamma_{s+r-1} \\ \vdots & & \vdots \\ \Gamma_{s+n-1} & \cdots & \Gamma_{s+r+n-2} \\ \Gamma_{s+n}^{(k)} & \cdots & \Gamma_{s+r+n-1}^{(k)} \end{vmatrix}^{-1}. \tag{3}$$

An other consequence is the fact that, for any s, the family $(P_r^s)_r$ is $1/d$ orthogonal [4], and satisfy a recurrence relation

$$P_{r+1}^s(x) = (x - \beta_r^s) P_r^s(x) - \sum_{\mu=1}^{d} \gamma_\mu^s P_{r-\mu}^s(x). \tag{D}$$

We will now study the consequences of (R) and (D) in order to find some algorithms to compute the P_r^s recursively.

In the first section, we will find a relation similar to the generalized MNA-algorithm [2]. In the second section we will give a vector QD-algorithm. In the last section we will define linear functionals $(C^\alpha)_{\alpha=1,\ldots,d}$ from one family P_r that verify (D), in such a way that orthogonal properties (R) are fullfilled.

1. Recursive computation of the P_r^s

In this section we will consider a different normalisation for the polynomials P_r^s:

$$P_r'^s = (-1)^r H_r^s(x) \begin{vmatrix} 1 & \cdots & 1 \\ \Gamma_s & \cdots & \Gamma_{s+r} \\ \vdots & & \vdots \\ \Gamma_{s+n}^{(k)} & \cdots & \Gamma_{s+r+n}^{(k)} \end{vmatrix}^{-1} = H_r^s(x)/H_r^s(1).$$

We get $P_r'^s(1) = 1$, instead of P_r^s monic.

For any k, let us define g_k:

$$k = hd + \beta, \quad g_k(i) = c^{\beta+1}(x^{i+h}).$$

With this notation, we get

$$\Gamma_s = \left(C_s^1 \dots C_s^d\right)^{\mathrm{T}} = \left(g_0(s),\ g_1(x), \dots, g_{d-1}(s)\right)^{\mathrm{T}},$$
$$P_r'^s = |(x),\ g_0(x), \dots, g_{r-1}(s)\,|/|1,\ g_0(s), \dots, g_{r-1}(s)\,|.$$

Theorem 1.1. *Let*

$$H_{r-1,i}^s = |\,g_i(s),\ g_0(s), \dots, g_{r-1}(s)\,|, \qquad h_{r-1,i}^s = H_{r-1,i}^s/H_r^s(1),$$

then it follows:

$$\begin{cases} P_r'^s(x) = -\left(h_{r-2,r-1}^{s+1} \cdot P_{r-1}'^s - x \cdot h_{r-2,r-1}^s \cdot P_{r-1}'^{s+1}\right)/\Delta_s h_{r-2,r-1}^s, \\ h_{r-1,i}^s = -\left(h_{r-2,r-1}^{s+1} \cdot h_{r-2,i}^s - h_{r-2,r-1}^s \cdot h_{r-2,i}^{s+1}\right)/\Delta_s h_{r-2,r-1}^s, \\ h_{-1,i}^s = g_i(s), \qquad P_0'^s = 1. \end{cases}$$

Δ_s *mean the classical forward difference with respect to the index s.*

Proof. Using Sylvester's identity for $(-1)^r H_r^s(x)$ and for $H_{r-1,i}^s$ gives

$$H_r'^s(x) = (-1)^r H_r^s(x) = |(x),\ g_0(s), \dots, g_{r-1}(s)\,|,$$
$$H_{r-1}^{s+1} H_r'^s(x) = H_r^{s+1} \cdot H_{r-1}'^s(x) - x H_r^s H_{r-1}'^{s+1}(x),$$
$$H_{r-1}^{s+1} H_{r-1,i}^s = H_r^{s+1} \cdot H_{r-2,i}^s - H_r^s H_{r-2,i}^{s+1}, \quad i \geqslant r \geqslant 1,$$
$$\begin{aligned} P_r'^s(x) &= H_r^s(x)/H_r^s(1) \\ &= \left(-H_r^{s+1} \cdot H_{r-1}^s(x) + x H_r^s \cdot H_{r+1}^{s+1}(x)\right)/\left(-H_r^{s+1} \cdot H_{r-1}^s(1) + H_r^s \cdot H_{r-1}^{s+1}(1)\right). \end{aligned}$$

Dividing by $H_{r-1}^s(1) \cdot H_{r-1}^{s+1}(1)$, it follows:

$$H_r^s = (-1)^{r-1} H_{r-2,r-1}^s, \qquad h_{r-1,i}^s = H_{r-1,i}^s/H_r^s(1)$$
$$P_r'^s(x) = \left(-h_{r-2,r-1}^{s+1} P_{r-1}'^s(x) + x h_{r-2,r-1}^s P_{r-1}'^{s+1}(x)\right)/\Delta_s h_{r-2,r-1}^s,$$
$$h_{r-1,i}^s = \left(-h_{r-2,r-1}^{s+1} h_{r-2,i}^s + h_{r-2,r-1}^s h_{r-2,i}^{s+1}\right)/\Delta_s h_{r-2,r-1}^s, \quad i \geqslant r \geqslant 1.$$

The initial conditions are consequences of the definitions. □

This result can be generalized in the following way.

Theorem 1.2.

$$P_{r+m}'^s = (-1)^m \begin{vmatrix} P_r'^s & x P_r'^{s+1} & \cdots & x^n P_r'^{s+m} \\ h_{r-i,r}^s & h_{r-1,r}^{s+1} & \cdots & h_{r-1,r}^{s+m} \\ \vdots & & & \vdots \\ h_{r-1,r+m+1}^s & & \cdots & h_{r-1,r+m-1}^{s+m} \end{vmatrix} \times \begin{vmatrix} 1 & \cdots & 1 \\ h_{r-1,r}^s & \cdots & h_{r-1,r}^{s+m} \\ \vdots & & \vdots \\ h_{r-1,r+m-1}^s & \cdots & h_{r-1,r+m-1}^{s+m} \end{vmatrix}^{-1}.$$

The quantities $h^s_{r,i}$ *satisfy the same relation, with* $(h^s_{r-1,i}, \dots, h^{s+m}_{r-1,i})$ *as the first row of the numerator.*

Proof. For $m = 1$ the relations are those of Theorem 1.1. They are supposed to be true until m. For $m + 1$, we get

$$N(P'^s_{r+m}) = \text{numerator of } P^s_{r+m}, \qquad D(P'^{s+1}_{r+m}) = \text{denominator of } P'^s_{r+m},$$

$$N = \left|(x^h P'^{s+h}_r),\ h^s_{r-1,r}, \dots, h^s_{r-1,r+m}\right|, \qquad D = |1,\ h^s_{r-1,r}, \dots, h^s_{r-1,r+m}|.$$

Applying Sylvester's identity to N and D, we have:

$$N \cdot \left|h^{s+1}_{r-1,r}, \dots, h^{s+1}_{r-1,r+m-1}\right| = N(P'^s_{r+m}) \cdot \left|h^{s+1}_{r-1,r}, \dots, h^{s+1}_{r-1,r+m}\right| - xN(P'^{s+1}_{r+m}) \cdot \left|h^s_{r-1,r}, \dots, h^s_{r-1,r+m}\right|.$$

$$D \cdot \left|h^{s+1}_{r-1,r}, \dots, h^{s+1}_{r-1,r+m-1}\right| = D(P'^s_{r+m}) \cdot \left|h^{s+1}_{r-1,r}, \dots, h^{s+1}_{r-1,r+m}\right| - D(P'^{s+1}_{r+m}) \cdot \left|h^{s+1}_{r-1,r}, \dots, h^{s+1}_{r-1,r+m}\right|.$$

Dividing by $D(P'^s_{r+m}) \cdot D(P'^{s+1}_{r+m})$, we get

$$N/D = \left(P'^s_{r+m} A(s+1) - P'^{s+1}_{r+m} A(s)\right) / (A(s+1) - A(s)),$$

$$A(s) = |h^s_{r-1,r}, \dots, h^s_{r-1,r+m}| / |1,\ h^s_{r-1,r}, \dots, h^s_{r-1,r+m-1}|,$$

$$A(s) = (-1)^m h^s_{r+m-1,r+m} \Rightarrow N/D = P'^s_{r+m+1}.$$

The formula concerning $h^s_{r+m,i}$, is proved in the same way by developing

$$N' = |h^s_{r-1,i},\ h^s_{r-1,r}, \dots, h^s_{r-1,r+m}|$$

instead of N. □

2. Vector QD-algorithm [3]

Let us now come back to the monic polynomials $P^s_r = H^s_r(x)/H^s_r$.

Theorem 2.1.

$$\begin{cases} P^s_{r+1}(x) = xP^{s+1}_r(x) - q^s_{r+1} P^s_r(x), & r \geqslant 0, \quad s \geqslant 0, \\ q^s_{r+1} = H^{s+1}_{r+1} \cdot H^s_r / H^s_{r+1} H^{s+1}_r. \end{cases} \tag{5}$$

Proof. Applying Sylvester's identity to $H^s_{r+1}(x)$, and dividing by $H^s_{r+1} H^{s+1}_r$, we get

$$H^{s+1}_r \cdot H^s_{r+1}(x) = -H^{s+1}_{r+1} \cdot H^s_r(x) + xH^s_{r+1} \cdot H^{s+1}_r(x),$$

$$P^s_{r+1}(x) = xP^{s+1}_r(x) - \left(H^{s+1}_{r+1} H^s_r / H^s_{r+1} H^{s+1}_r\right) \cdot P^s_r(x).$$

This relation is the same as in the scalar case $d = 1$. □

Theorem 2.2.

$$P_r^{s+1}(x) - P_r^s(x) = -\sum_{i=r-d}^{r-1} e_{r,i}^s P_i^{s+1}(x). \tag{6}$$

Proof. The polynomials P_r^s are monic, so $P_r^{s+1} - P_r^s$ is a polynomial of degree $r-1$, and can be expanded as

$$P_r^{s+1}(x) - P_r^s(x) = \sum_0^{r-1} \lambda_i P_i^{r+1}(x),$$

$$\begin{cases} \Gamma\left(x^j, \left(P_r^{s+1} - P_r^s\right)\right) = 0, & j = s+1, \ldots, s+n-1, \quad r = nd+k; \\ C^\alpha\left(r^{s+n}, \left(P_r^{s+1} - P_r^s\right)\right) = 0, & \alpha = 1, \ldots, k. \end{cases}$$

$$\Rightarrow \lambda_0 = \cdots = \lambda_{(n-1)d+k-1} = 0.$$

The $(\lambda_i)_{r-d,\ldots,r-1}$ satisfy a linear system

$$\begin{cases} \lambda_{r-d} C^{k+1}\left(x^{s+n} P_{r-d}^{s+1}\right) = -C^{k+1}\left(x^{s+n} P_r^s\right), \\ \vdots \\ \lambda_{r-d} C^d\left(x^{s+n} P_{r-d}^{s+1}\right) + \cdots + \lambda_{nd-1} C^d\left(x^{s+n} P_{nd-1}^{s+1}\right) = -C^d\left(x^{s+n} P_r^s\right), \\ \vdots \\ \lambda_{r-d} C^k\left(x^{s+n+1} P_{r-d}^{s+1}\right) + \cdots + \lambda_{r-1} C^k\left(x^{s+n+1} P_{r-1}^{s+1}\right) = -C^k\left(x^{s+n+1} P_r^s\right), \end{cases}$$

whose determinant is the product:

$$\frac{H_{r-d+1}^{s+1}}{H_{r-d}^{s+1}} \cdot \frac{H_{r-d+2}^{s+1}}{H_{r-d+1}^{s+1}} \cdots \frac{H_r^{s+1}}{H_{r-1}^{s+1}} = \frac{H_r^{s+1}}{H_{r-d}^{s+1}},$$

and whose second member is

$$-\left[\frac{H_{r-1,r}^s}{H_r^s}, \ldots, \frac{H_{r-1,r+i}^s}{H_r}, \ldots, \frac{H_{r-1,r+d-1}^s}{H_r^s}\right] = -\left(h_{r-1,r}^s, \ldots, h_{r-1,r+d-1}^s\right).$$

Let A^k the matrix of the system:

$$A = \begin{bmatrix} h_{r-d-1,r-d}^{s+1} & & 0 \\ \vdots & \ddots & \\ h_{r-d-1,r-1}^{s+1} & \cdots & h_{r-1,r}^{s+1} \end{bmatrix}.$$

The relation (6) can be written

$$P_r^{s+1} - P_r^s = \begin{bmatrix} 0 & P_{r-d}^{s+1} & \cdots & P_{r-1}^{s+1} \\ h_{r-1,r}^s & & & \\ \vdots & & A & \\ h_{r-1,r-d-1}^s & & & \end{bmatrix} \cdot \frac{H_{r-d}^{s+1}}{H_r^{s+1}} = -\sum_{r-d}^{r-1} e_{r,i}^s P_i^{s+1}.$$

Considering equations (D), (5), (6) it is clear that any two of these three imply the third one. □

Theorem 2.3. *Vector QD-algorithm*:

$$\begin{cases} q_{r+1}^{s+1} + e_{r,r-1}^{s+1} = q_{r+1}^{s} + e_{r+1,r}^{s}, & (7a) \\ e_{r,i}^{s+1} q_{i+1}^{s+1} + e_{r,i-1}^{s+1} = e_{r,i}^{s} q_{r+1}^{s} + e_{r+1,i}^{s}, \quad i = r-d+1, \ldots, r-1, \quad d > 1, & (7b) \\ e_{r,r-d}^{s+1} q_{r-d+1}^{s+1} = e_{r,r-d}^{s} q_{r+1}^{s}, \quad r \geqslant d. & (7c) \end{cases}$$

Proof. From the relations (5) and (6), the relation (D) of polynomials P_r^s (s fixed) follows:

$$P_{r+1}^{s} = xP_r^{s+1} - q_{r+1}^{s} P_r^{s},$$

$$P_r^{s+1} = P_r^s - \sum_{r-d}^{r-1} e_{r,i}^{s} P_i^{s+1},$$

$$P_{r+1}^{s} = x\left[P_r^s - \sum_i e_{r,i}^{s} P_i^{s+1}\right] - q_{r+1}^{s} P_r^{s} = (x - q_{r+1}^{s}) P_r^s - \sum_i e_{ri}^{s} x P_i^{s+1}$$

$$= (x - q_{r+1}^{s}) P_r^s - \sum_i e_{ri}^{s} (P_{i+1}^{s} + q_{i+1}^{s} P_i^{s}),$$

$$P_{r+1}^{s} = (x - q_{r+1}^{s} - e_{r,r-1}^{s}) P_r^s - \sum_{r-d+1}^{r-1} (e_{r,i-1}^{s} + e_{ri}^{s} q_{i+1}^{s}) P_i^s - e_{r,r-d}^{s} q_{r-d+1}^{s} P_{r-d}^{s}. \tag{6a}$$

In the same way, we get for P_{r+1}^{s+1}

$$P_{r+1}^{s+1} = P_{r+1}^{s} - \sum_{r-d+1}^{r} e_{r+1,i}^{s} p_i^{s+1}$$

$$= xP_r^{s+1} - q_{r+1}^{s}\left(P_r^{s+1} + \sum_{r-d}^{r-1} e_{ri}^{s} P_i^{s+1}\right) - \sum_{r-d+1}^{r} e_{r+1,i}^{s} P_i^{s+1},$$

$$P_{r+1}^{s+1} = (x - q_{r+1}^{s} - e_{r+i,r}^{s}) P_r^{s+1} - \sum_{r-d+1}^{r-1} \{e_{ri}^{s} q_{r+1}^{s} + e_{r+1,i}^{s}\} P_i^{s+1} - e_{r,r-d}^{s} q_{r+r}^{s} P_{r-d}^{s+1}. \tag{6b}$$

Identifying (6b) and (6a) for $s+1$, the relations (7) follow. □

Let $E_r^s = (e_{ri}^s)_{i=r-d,\ldots,r-1}$. These quantities are displayed in a array as for the scalar QD-algorithm.

$$\begin{array}{cccc} q_{r-1}^{s} & & q_r^{s-1} & \\ & E_{r-1}^{s} & & E_r^{s-1} \\ q_{r-1}^{s+1} & & q_r^{s} & \\ & E_{r-1}^{s+1} & & E_r^{s} \\ q_{r-1}^{s+2} & & q_r^{s+1} & \end{array}$$

The computations will go on, nearly as in the scalar case:
– if all the columns are known up to the index r, q_{r+1}^{s} will be obtained by equation (7c), and then E_{r+1}^{s} by equations (7b) and (7a).

– If a diagonal (index s) and the columns $q_k, \ldots, q_{k+d}$ are known ($k \geqslant 0$), E_i^{s+1} will be given by (7c) and (7b), and then q_{i+1}^{s+1} by (7a) for $i \geqslant k+d$ and so on.
– If a diagonal (index s) and the column E_{k+d} are known, $q_k^{s+1} \ldots q_{k+d}^{s+1}$ ($k \geqslant 0$) will be obtained by (7c) and (7b), and then q_{k+d+1}^{s+1} by (7a), and finally we have E_{k+d+1}^{s+1} ((7c) and (7b)) and $q_{k+d+2}^{s+1}, \ldots$, that is the diagonal of index $s+1$.

Let us now have a look at the initial conditions of the algorithm. For $r < d$, only (7a) and (7b) can be obtained, and thus $E_k^s = (e_{k,i}^s)_{i=0,k-1}$ can be computed from q_i and for, $i \leqslant k \leqslant d$, but not the contrary.

Every quantity with a negative index is zero, and we compute the columns $q_1, \ldots, q_d$, which define completely E_i and q_i.

$$\begin{cases} E_0^n = 0, \\ q_1^s = \dfrac{c_{s+1}^1}{c_s^1}, \qquad q_{i+1}^s = \dfrac{H_{i+1}^{s+1} H_i^s}{H_{i+1}^s H_i^{s+1}}, \qquad i = 0, \ldots, d-1. \end{cases}$$

It must be noticed that in practical cases the determinants H_i^s and H_{i+1}^s are easy to compute since, i is less than $d-1$, and d must not be very large: the improvement on the degree of approximation from Padé type approximations to Padé approximation is the integer part of r/d for a rational function (s/r).

The computation of H_k^s for $k < i$ does not need the functional C^i, so it is impossible to have less initial conditions: C_k^i are involved for the first time in the computation of the ith column.

3. Relation between polynomials verifying (D) and vector orthogonality (R)

The polynomial P_r^s have been obtained from the relations (R) called relations of vector orthogonality,

$$\text{(R)} \qquad \begin{cases} C^\alpha\left(P_{nd+k}^s \cdot x^i\right) = 0, \qquad i = s, \ldots, s+n-1, \quad \alpha = 1, \ldots, d, \\ C^\alpha\left(P_{nd+k}^s \cdot x^{n+s}\right) = 0, \quad \alpha = 1, \ldots, k. \end{cases}$$

P_r^s have been obtained through (R), from $C^1, \ldots, C^d$, defined by

$$C^\alpha(x^i) = C_i^\alpha, \quad i \geqslant 0.$$

We will now prove that it is possible to build $C^1, \ldots, C^d$ from one family verifying relation (D) (called D-family).

Theorem 3.1. *Let $(P_r)_{r \geqslant 0}$ be a D-family, there exists (P_r^s), $r \geqslant 0$, $s \geqslant 0$, which satisfy*

$$\begin{cases} P_r^0 = P_r, \quad r \geqslant 0, \\ (P_r^s)_{r \geqslant 0} \text{ is a D-family}, \\ (P_r^s)_r \text{ and } \left(P_r^{s+1}\right)_r \text{ are linked together by the QD-algorithm.} \end{cases}$$

Theorem 3.2. *Let $(P_r)_{r\geqslant 0}$ being as above, there exist $C^1,\dots,C^d$ linear functionals from $\mathbb{C}[x]$ to $\mathbb{C}$ such that relation* (R) *are fullfilled. Each C^α is uniquely determined by $C_0^\alpha,\dots,C_{\alpha-1}^\alpha$.*

Proof of Theorem 3.1.

First step: Construction of q_r^0, $E_r^0=(e_{r,r-1}^0,\dots,e_{r,r-d}^0)$. The upper index 0 will be omitted in that part.

$$P_{r+1}=(x-\beta_{r+1})P_r-\sum_{r-d}^{r-1}\gamma_\mu P_\mu;$$

$$\text{(i)}\qquad \beta_{r+1}=q_{r+1}+e_{r,r-1},$$

$$\text{(ii)}\qquad \gamma_\mu=e_{r,\mu-1}+e_{r\mu}q_{\mu+1},\qquad \mu=r-d+1,\dots,r-1.$$

$$\text{(iii)}\qquad \gamma_{r-d}=e_{r,r-d}\cdot q_{r+1}.$$

If P_{r+1}, $q_1,\dots,q_r$ are known, $e_{r,r-d}$ is given by (3), $e_{r\mu}$ ($\mu=r-d+1,\dots,r-1$) by (2) and q_{r+1} by (1). If $r<d$, the recurrence relation can still be written as the decomposition of P_{r+1} on the basis $P_0,\dots,P_r$, xP_r.

$$\begin{aligned}
&P_0=1,\\
&P_1=(x-\beta_1)P_0, && q_1=\beta_1,\\
&P_2=(x-\beta_2)P_1-\gamma_0^2P_0, && q_2+e_{1,0}=\beta_2,\\
& && q_1e_{1,0}=\gamma_0^2\Rightarrow e_{1,0},\ q_2.
\end{aligned}$$

The sequences $(q_r^0)_r$, and $(E_r^0)_r$ are completely defined for $r\geqslant 1$, by the polynomials P_r.

Second step: construction of P_r^s. Let us prove that $P_{r+1}+q_{r+1}P_r$ can be divided by x:

$$P_{r+1}+q_{r+1}P_r=x\cdot P_r-\sum_{r-d}^{r-1}\left(P_{\mu+1}+q_{\mu+1}P_\mu\right)e_{r\mu},$$

$$P_1+q_1P_0=x\cdot P_0.$$

So for any μ, $P_{\mu+1}+q_{\mu+1}P_\mu$ is divisible by x, and p_r^1 is defined by

$$P_{r+1}+q_{r+1}P_r=xP_r^1,\qquad r\geqslant 0,$$

$$P_0^1=P_0=1.$$

The polynomials P_r^1 form a (D)-family:

$$\begin{cases}P_{r+1}=(x-q_{r+1}-e_{r,r-1})P_r-\displaystyle\sum_{r-d+1}^{r-1}(e_{r\mu-1}+e_{r\mu}q_{\mu+1})P_\mu-e_{r,r-d}q_{r-d+1}P_{r-d},\\ P_{r+1}=xP_r^1+q_{r+1}P_r.\end{cases}$$

$$xP_r^1=xP_r-\sum_{r-d}^{r-1}e_{r\mu}\left(P_{\mu+1}+q_{\mu+1}P_\mu\right)=x\left(P_r-\sum_{r-d}^{r-1}e_{r\mu}P_\mu^1\right)$$

$$\Rightarrow P_r^1=P_r-\sum_{r-d}^{r-1}e_{r\mu}P_\mu^1,$$

$$P_{r+1}^1 = P_{r+1} - \sum_{r+1-d}^{r} e_{r+\mu,\mu} P^1,$$

$$= (x - q_{r+1} - e_{r,r-1}) P_r - \sum_{r-d}^{r-1} \cdots - e_{r,r-d} q_{r+1} P_{r-d} - \sum_{r-d+1}^{r} e_{r+1,\mu} P_\mu^1$$

$$= (x - q_{r+1}) \left(P_1^r + \sum_{r-d}^{r-1} e_{r\mu} P_\mu^1 \right) - \sum_{r-d}^{r-1} e_{r\mu} (P_{\mu+1} + q_{\mu+1} P_\mu) - \sum_{r-d+1}^{r} e_{r+1,\mu} P_\mu^1$$

$$\Rightarrow P_{r+1}^1 = (x - q_{r+1} - e_{r+1,r}) P_r^1 - \sum_{r-d+1}^{r-1} (e_{r\mu} q_{r+1} + e_{r+1,\mu}) P_\mu^1 - e_{r,r-d} q_{r+1} P_{r-d}^1.$$

Again from $(P_r^1)_{r \geqslant 0}$, we construct q_r^1 and E_r^1 by the vector QD-algorithm, then $(P_r^2)_{r \geqslant 0}$ and so on.

Proof of Theorem 3.2.

First step: $(C^\alpha)_{\alpha=1,\ldots,d}$ are defined by the following system:

$$\begin{cases} C^1(P_r) = 0, & r \geqslant 1, \\ C^\alpha(P_r) = 0, & r \geqslant \alpha, \\ C^d(P_r) = 0, & r \geqslant d, \end{cases}$$

$$P_r(x) = x^r + \sum_0^{r-1} a_{ri} x^i,$$

$$C^\alpha(P_r) = C_r^\alpha + \sum_0^{r-1} a_{ri} C_i^\alpha = 0, \quad r \geqslant \alpha.$$

When $C_0^\alpha \cdots C_{\alpha-1}^\alpha$ are given, all $(C_r^\alpha)_{r \geqslant \alpha}$ are uniquely defined. As usual, if $d = 1$, C^1 is defined from C_0^1; all the solutions are of the form λC^1.

Second step: (P_n) satisfy relations (R). We have to prove, with $0 \leqslant k < d$,

$$\begin{cases} c^\alpha(P_{nd+k} \cdot x^i) = 0, & i = 0, \ldots, n-1, \quad \alpha = 1, \ldots, d, \\ c^\alpha(P_{nd+k} \cdot x^n) = 0, & \alpha = 1, \ldots, k; \end{cases}$$

$$x P_r = P_{r+1} + \beta_{r+1} P_r + \sum_{\mu=r-d}^{r-1} \gamma_\mu P_\mu.$$

So $C^\alpha(x P_r) = 0$ is equivalent to $C^\alpha(P_{r-d}) = 0$, i.e. $r \geqslant d + \alpha$. Again we have

$$C^\alpha(P_r x^i) = 0 \Leftrightarrow C^\alpha(P_{r-id}) = 0 \quad \text{i.e. } r \geqslant id + \alpha$$

and the result follows.

Third step: (P_r^s) satisfy relations (R). We have to prove, always with $0 \leqslant k < d$,

$$\begin{cases} c^\alpha(P_{nd+k} \cdot x^i) = 0, & i = s, \ldots, s+n-1, \quad \alpha = 1, \ldots, d, \\ c^\alpha(P_{nd+k}^s \cdot x^{n+s}) = 0, & \alpha = 1, \ldots, k, \end{cases}$$

$$x P_r^{s+1} = P_{r+1}^s + q_{r+1}^s P_r^s,$$

$$C^\alpha(x^i P_r^{s+1}) = C^\alpha(x^{i-1} P_{r+1}^s) + q_{r+1}^s C^\alpha(x^{i-1} P_r^s)$$
$$= 0, \quad i - 1 = s, \ldots, s+n-1 \quad \text{or} \quad i = s+2, \ldots, s+n.$$

It is the same for the second assertion.

Finally the functionals C^α, $\alpha = 1, \ldots, d$, and the polynomials P_r^s, constructed from the family (P_r), satisfy the vector orthogonality relations. Each functional C^α is defined by α constants, so considering $\Gamma = (C^1, \ldots, C^d)$, the result can be put in the following form: the set of functionals Γ associated with a family P_r which satisfy a recurrence relation of type (D), is a vector subspace of $L(\mathbb{C}[[X]], \mathbb{C}^d)$ of dimension $d!$ This result is a generalization of the Shohat-Favard theorem: there is a unique functional associated with a family of polynomials satisfying a three term recurrence formula.

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